Twisted $\mathcal{N} = 4$ Super Yang-Mills Theory in Ω-background

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1. Introduction

Ω-background deformation [Moore-Nekrasov-Shatashvili]

$$\Omega\text{-background for } \mathcal{N}=2 \text{ theory} \qquad (\leftarrow \text{6D } \mathcal{N}=(1,0) \text{ theory})$$

$$= \underbrace{\text{6D curved background}}_{\text{for deformation}} + \underbrace{\text{SU(2) R-symmetry Wilson line}}_{\text{to preserve SUSY}}$$

- 6D Ω-background metric (on $\mathbb{R}^4 \times \mathbf{T}^2$)

$$ds^2 = 2d\bar{z}dz + (dx^\mu + \Omega^{\mu\nu}x_\nu d\bar{z} + \bar{\Omega}^{\mu\nu}x_\nu dz)^2.$$

Here constant antisymmetric matrix $\Omega^{\mu\nu}$ and $\bar{\Omega}^{\mu\nu}$ commute with each other.

Hence they can be taken of the form

$$\Omega^{\mu\nu} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & \epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{\mu\nu} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\bar{\epsilon}_2 \\ 0 & 0 & \bar{\epsilon}_2 & 0 \end{pmatrix}.$$

- $\bar{Q}_0 \to \bar{Q}$, $\quad \bar{Q}^2 =$ (gauge transf.) $+$ ($\mathrm{U}(1)^2$ rotation by ϵ_1, ϵ_2)
- $S_0 = \bar{Q}_0 \Xi_0 \to S = \bar{Q}\Xi = \bar{Q}(\Xi_0 + \cdots)$

equivariant localization $\Longrightarrow$ exact computation of path integral

- Nekrasov partition function

Generalization of Ω-background for $\mathcal{N}=4$ super Yang-Mills theory [Ito-H.N.-Saka-Sasaki], [Ito-H.N.-Sasaki 2012]

generalized Ω-background
$=$ 10D curved background $+$ SU(4) R-symmetry Wilson line

In 10D, SU(4) $=$ SO(6) R-symmetry is the subgroup of local Lorentz symmetry (No R-symmetry in 10D SYM).

$\Rightarrow$ contribution to spin (and Affine) connection ($\sim$ torsion)

$\Rightarrow$ deformation of parallel (or Killing) spinor equation

$$\nabla_{\mathcal{M}}\zeta = 0 \quad \longrightarrow \quad \widehat{\nabla}_{\mathcal{M}}\zeta = \left(\nabla_{\mathcal{M}} + \frac{1}{4}\mathcal{A}_{\mathcal{M},NP}\Gamma^{NP}\right)\zeta = 0.$$

Classification of parallel spinors $\rightarrow$ supersymmetry in the theory

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4. Summary

2. Ω-background and SUSY condition

Ω-background for $\mathcal{N}=2$ theory
$=$ 6D curved background $+$ SU(2) R-symmetry Wilson line

generalized Ω-background
$=$ 10D curved background $+$ SU(4) R-symmetry Wilson line

(con)torsion

• 10D metric for generalized Ω-background (on $\mathbb{R}^4 \times \mathbb{T}^6$)

$$ds^2_{10\mathrm{D}} = (dx^\mu + \Omega^\mu_a dx^a)^2 + dx^a dx^a, \quad \Omega^\mu_a = \Omega^\mu{}_{\nu a} x^\nu.$$

Here $\Omega_{\mu\nu a} = -\Omega_{\nu\mu a}$ are constant and commute with each other.
(μ, ν, . . . : 4D indices, a, b, . . . : 6D indices)

commuting $\Omega_{\mu\nu a} \longrightarrow \Omega_{\mu\nu a} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & \epsilon_{1a} & 0 & 0 \\ -\epsilon_{1a} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_{2a} \\ 0 & 0 & \epsilon_{2a} & 0 \end{pmatrix}.$

R-symmetry Wilson line: $\mathcal{A}_{bc} = \mathcal{A}_{a,bc}\, dx^a$, $\mathcal{A}_{a,bc}$: constant

Action

$$S_\Omega = \int d^4x \, \frac{1}{\kappa g^2} \mathrm{Tr}\Big[\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu\varphi_a - F_{\mu\nu}\Omega^\nu_a)^2 + \frac{1}{4}\Big[[\varphi_a, \varphi_b] + i\Omega^\mu_a D_\mu\varphi_b - i\Omega^\mu_b D_\mu\varphi_a - i\Omega^\mu_a\Omega^\nu_b F_{\mu\nu} - i\big(\mathcal{A}_{a,b}{}^c - \mathcal{A}_{b,a}{}^c\big)\varphi_c\Big]^2 + (\text{fermions})\Big].$$

• parallel spinor condition and topological twist

$$\partial_\mu \zeta = 0, \quad \frac{1}{4}(\Omega_{\mu\nu a}\Gamma^{\mu\nu} + \mathcal{A}_{a,bc}\Gamma^{bc})\zeta = 0.$$

$\Longrightarrow$ 4D and 6D rotation should be canceled.

$$\Omega_{\mu\nu a} \in \mathsf{U}(1)_L \times \mathsf{U}(1)_R \subset \mathsf{SU}(2)_L \times \mathsf{SU}(2)_R.$$

In order to cancel 4D rotation, we restrict the contorsion s.t.

$$\mathcal{A}_{a,bc} \in \mathsf{U}(1)_{L'} \times \mathsf{U}(1)_{R'} \subset \mathsf{SU}(2)_{L'} \times \mathsf{SU}(2)_{R'} \subset \mathsf{SU}(4).$$

cancellation of U(1) charges $\Longrightarrow$ topological twist

SUSY condition can be solved for each topological twist.
(We assume generic ϵ_1, ϵ_2.)

- half twist $(\mathrm{SU}(2)_R \sim \mathrm{SU}(2)_{R'})$ $\Rightarrow$ $\mathcal{N}=2^*$ theory

  parameters : $\epsilon_1,\ \epsilon_2,\ \bar{\epsilon}_1,\ \bar{\epsilon}_2,\ m,\ \bar{m}$ (m, $\bar{m}$: $\mathcal{N}=2^*$ mass)

  one scalar and one tensor supercharges are preserved.

- Vafa-Witten twist $(\mathrm{SU}(2)_R \sim \mathrm{diag}(\mathrm{SU}(2)_{L'} \times \mathrm{SU}(2)_{R'}))$

  parameters : $\epsilon_1,\ \epsilon_2,\ \bar{\epsilon}_1,\ \bar{\epsilon}_2,\ \hat{\epsilon}_1,\ \hat{\epsilon}_2,\ \check{\epsilon}_1,\ \check{\epsilon}_2$

  two scalar and two tensor supercharges are preserved.

- Marcus twist $(\mathrm{SU}(2)_L \sim \mathrm{SU}(2)_{L'}\,,\ \mathrm{SU}(2)_R \sim \mathrm{SU}(2)_{R'})$

  parameters : $\epsilon_1,\ \epsilon_2,\ \bar{\epsilon}_1,\ \bar{\epsilon}_2$ (special case of half twist)

  two scalar and two tensor supercharges are preserved.

3. Off-shell scalar supersymmetry

(a) half twist ($\mathcal{N} = 2^*$ theory)

$$A_\mu \to A_\mu, \qquad \varphi_a \to (\varphi, \bar{\varphi}, \varphi^{\dot{\alpha}\hat{A}}),$$

$$\Lambda^A_\alpha \to (\Lambda_\mu, \Lambda^{\hat{A}}_\alpha), \qquad \bar{\Lambda}^{\dot{\alpha}}_A \to (\bar{\Lambda}, \bar{\Lambda}^-_{\mu\nu}, \bar{\Lambda}^{\dot{\alpha}}_{\hat{A}})$$

$\bar{\Lambda}^-_{\mu\nu}$ and $\Lambda^{\hat{A}}_\alpha$ need auxiliary fields for off-shell closure.

auxiliary fields

$$\bar{Q}\bar{\Lambda}_{\mu\nu} = -2F^-_{\mu\nu} - i(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}[\varphi^{\dot{\alpha}\hat{A}}, \bar{\varphi}_{\hat{A}\dot{\beta}}] + 2D^-_{\mu\nu},$$

$$\bar{Q}\Lambda^{\hat{A}}_\alpha = \sqrt{2}(\sigma^\mu)_{\alpha\dot{\alpha}} D_\mu \varphi^{\dot{\alpha}\hat{A}} + 2K^{\hat{A}}_\alpha.$$

Algebra of scalar supercharges

$$\bar{Q}^2 = 2\sqrt{2}\big(\,\delta_{\rm gauge}(\varphi) + \delta_{\rm Lorentz}(\epsilon_1,\,\epsilon_2) + \delta_{\rm flavor}(m)\,\big),$$

$\delta_{\rm gauge}(\varphi)$: gauge transformation by φ.

$\delta_{\rm Lorentz}(\epsilon_1,\,\epsilon_2)$: $\mathsf{U}(1)_L \times \mathsf{U}(1)_{R+R'}$ rotation by parameter $\epsilon_1,\,\epsilon_2$.

$\delta_{\rm flavor}(m)$: $\mathsf{U}(1)_{L'}$ rotation by parameter m.

The action S is written as $\bar{Q}$-exact form up to the topological term:

$$S = \bar{Q}\Xi + \int d^4x \frac{1}{\kappa g^2} {\rm Tr}\left[\frac{1}{4} F_{\mu\nu}\tilde{F}^{\mu\nu}\right].$$

(b) Vafa-Witten twist [Vafa-Witten], [Dijkgraaf-Moore], etc.

$$A_\mu \to A_\mu, \qquad \varphi_a \to (\varphi, \bar\varphi, \hat\varphi, \hat\varphi^-_{\mu\nu}),$$
$$\Lambda^A_\alpha \to (\Lambda_\mu, \hat\Lambda_\mu), \qquad \bar\Lambda^{\dot\alpha}_A \to (\bar\Lambda, \bar\Lambda^-_{\mu\nu}, \hat{\bar\Lambda}, \hat{\bar\Lambda}^-_{\mu\nu})$$

Λ_μ, $\hat\Lambda_\mu$, $\bar\Lambda^-_{\mu\nu}$ and $\hat{\bar\Lambda}^-_{\mu\nu}$ need auxiliary fields for off-shell closure.

$\Longrightarrow$ auxiliary fields: K_μ, $D^-_{\mu\nu}$

Algebra of scalar supercharges

$$\bar Q^2 = 2\sqrt2\left(\,\delta_{\rm gauge}(\varphi) + \delta_{\rm Lorentz}(\epsilon_1,\ \epsilon_2)\,\right),$$
$$\hat{\bar Q}^2 = -2\sqrt2\left(\,\delta_{\rm gauge}(\bar\varphi) + \delta_{\rm Lorentz}(\bar\epsilon_1,\ \bar\epsilon_2)\,\right),$$
$$\{\bar Q, \hat{\bar Q}\} = 2\sqrt2\left(\,\delta_{\rm gauge}(\hat\varphi) + \delta_{\rm Lorentz}(\hat\epsilon_1,\ \hat\epsilon_2)\,\right).$$

The action is written in the exact form with respect to the two scalar supercharges simultaneously:

$$S = \bar{Q}\hat{\bar{Q}}\mathcal{F} + \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\left[\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu}\right].$$

Here

$$\mathcal{F} = \int d^4x \, \frac{1}{\kappa g^2}\mathrm{Tr}\Big[-\frac{1}{2\sqrt{2}}\hat{\varphi}^{\mu\nu}F^-_{\mu\nu} + \frac{1}{8}\bar{\Lambda}^{\mu\nu}\hat{\bar{\Lambda}}_{\mu\nu} + \frac{1}{8}\Lambda^{\mu}\Lambda_{\mu} - \frac{1}{8}\bar{\Lambda}\hat{\bar{\Lambda}} + \frac{i}{24\sqrt{2}}\hat{\varphi}^{\mu\nu}[\hat{\varphi}_{\mu}{}^{\lambda}, \hat{\varphi}_{\lambda\nu}]$$
$$+ \frac{1}{16\sqrt{2}}\hat{\varphi}^{\mu\nu}(\hat{\Omega}^{\rho,}{}_{\mu\sigma}D_{\rho}\hat{\varphi}_{\nu}{}^{\sigma} - \hat{\Omega}^{\rho,}{}_{\nu\sigma}D_{\rho}\hat{\varphi}_{\mu}{}^{\sigma} - i\hat{\Omega}_{\mu\nu,\rho\sigma}\hat{\varphi}^{\rho\sigma} + i\hat{\Omega}_{\rho\sigma,}{}^{\rho\sigma}\hat{\varphi}_{\mu\nu})$$
$$+ \frac{3}{2\sqrt{2}}\hat{\Omega}^{[\rho,\mu\nu]}\left(A_{[\mu}F_{\nu\rho]} - \frac{i}{3}A_{[\mu}A_{\nu}A_{\rho]}\right)\Big],$$

$$\hat{\Omega}^{\rho,}{}_{\mu\nu} = \hat{\Omega}^{\rho\sigma,}{}_{\mu\nu}\, x_{\sigma}, \quad \hat{\Omega}_{12,12} = -\hat{\Omega}_{12,34} = -\frac{1}{\sqrt{2}}\check{\epsilon}_1, \quad \hat{\Omega}_{34,12} = -\hat{\Omega}_{34,34} = \frac{1}{\sqrt{2}}\check{\epsilon}_2.$$

(c) Marcus twist (GL twist) [Marcus], [Kapustin-Witten]

$$A_\mu \to A_\mu, \qquad \varphi_a \to (\varphi, \bar{\varphi}, \varphi_\mu),$$
$$\Lambda_\alpha^A \to (\Lambda_\mu, \Lambda, \Lambda_{\mu\nu}^+), \qquad \bar{\Lambda}_A^{\dot{\alpha}} \to (\bar{\Lambda}, \bar{\Lambda}_{\mu\nu}^-, \bar{\Lambda}_\mu)$$

Λ, $\bar{\Lambda}$, $\Lambda_{\mu\nu}^+$ and $\bar{\Lambda}_{\mu\nu}^-$ need auxiliary fields for off-shell closure.

$\Longrightarrow$ auxiliary fields: K, $K_{\mu\nu}^+$, $D_{\mu\nu}^-$

On-shell algebra of scalar supercharges

$$Q^2 = \bar{Q}^2 = 2\sqrt{2}\big(\,\delta_{\rm gauge}(\varphi) + \delta_{\rm Lorentz}(\epsilon_1,\,\epsilon_2)\,\big),$$
$$\{Q, \bar{Q}\} = 0.$$

The first equation holds off-shell but the second does not hold off-shell on the fields $\Lambda_{\mu\nu}^+$ and $\bar{\Lambda}_{\mu\nu}^-$.

linear combination of scalar supercharges [Kapustin-Witten]

$$\mathcal{Q} = uQ + v\bar{Q}, \quad u, v \in \mathbb{C},$$

$$\mathcal{Q}^2 = 2\sqrt{2}(u^2 + v^2)\big(\,\delta_{\rm gauge}(\varphi) + \delta_{\rm Lorentz}(\epsilon_1,\, \epsilon_2)\,\big).$$

The above algebra holds off-shell.

$\mathcal{Q}$-exactness of the action

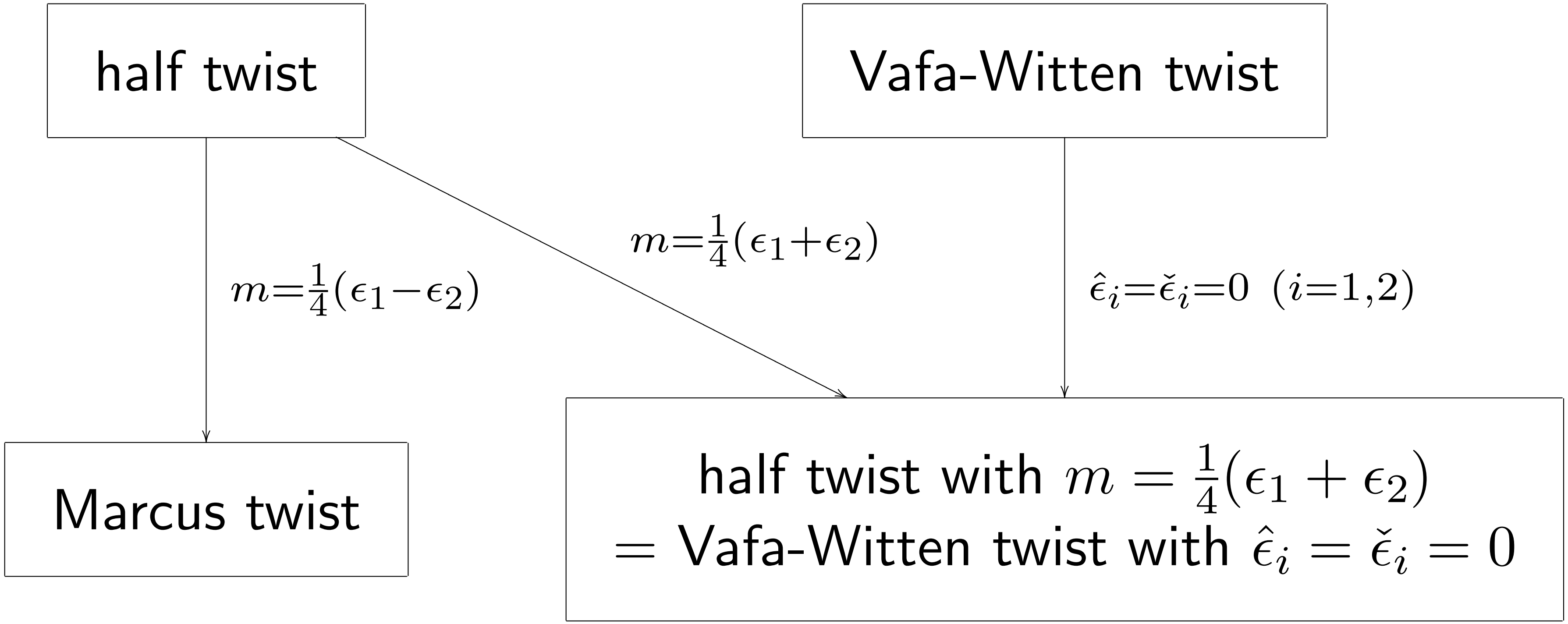
- $u^2 + v^2 \neq 0$ case

$$S = (\mathcal{Q}\text{-exact term}) + \int d^4x\, \frac{1}{\kappa g^2} \mathrm{Tr}\left[\frac{u^2 - v^2}{4(u^2 + v^2)} F_{\mu\nu}\tilde{F}^{\mu\nu}\right].$$

- $u^2 + v^2 = 0$ case

$$S = (\mathcal{Q}\text{-exact term}) + (\mathcal{Q}\text{-closed term, not } \mathcal{Q}\text{-exact}).$$

Relation among three topological twists



cf. [Pestun], [Pestun-Okuda], etc.

4. Summary

summary

1. We have constructed off-shell scalar supercharge(s) in Ω-deformed $\mathcal{N} = 4$ super Yang-Mills.

2. The action is written as the exact form with respect to the supercharge(s) up to the topological term except the case of Marcus twist with $u^2 + v^2 = 0$.

future work

- Nekrasov-Shatashvili limit, enhancement of (off-shell) SUSY
- extension to more complicated backgrounds
  [Festuccia-Seiberg], [Dumitrescu-Festuccia-Seiberg],
  [Hama-Hosomichi], [Klare-Zaffaroni] etc.
- auxiliary field formalism in 10D [Berkovits], etc.
- embedding to superstring/SUGRA
  R-R 3-form (in instanton (D(-1)) effective action)
  [Billo-Frau-Fucito-Lerda], [Ito-H.N.-Saka-Sasaki],
  [Antoniadis-Florakis-Hohenegger-Narain-Zein Assi], etc.
  (cf. [Hellerman-Orlando-Reffert], [Reffert], [Nakayama-Ooguri])